A piecewise affine approach to nonlinear performance

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Context

Since the 90s → important theoretical and methodological developments in control theory

- Emergence of robust control methods
- Appearance of efficient solvers → optimization problems

Systematically tackle a large number of engineering specifications for linear systems

Tight specifications → non negligible nonlinear effects

Engineering expertise (heuristics) → no *a priori* guarantees

Need to develop efficient methods for nonlinear **performance analysis**

Context

Extension of robust control to nonlinear systems

- Most of the literature concerns stability
  - $\hookrightarrow$ Not able to guarantee some qualitative specifications
- Proposal of incremental stability
- For linear systems: stability = incremental stability

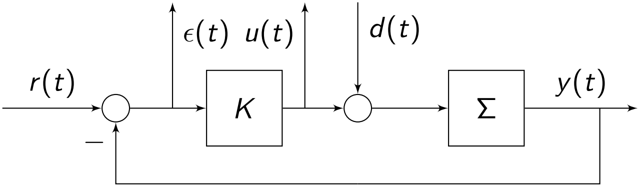
Complexity of necessary and sufficient conditions for nonlinear systems

$\hookrightarrow$ Development of relaxed sufficient conditions $\rightarrow$ conservatism

Reduce conservatism $\rightarrow$ piecewise affine representations

- Describe a wide range of nonlinear system dynamics
- Similar to linear systems $\rightarrow$ extension of efficient techniques

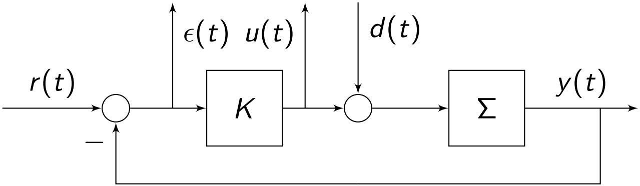
Typical control problem



Engineering specifications

- Stability
- Tracking
- Disturbance rejection
- Robustness

Typical control problem



Engineering specifications

- Stability
- Tracking
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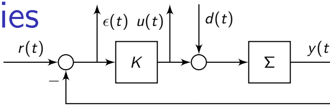
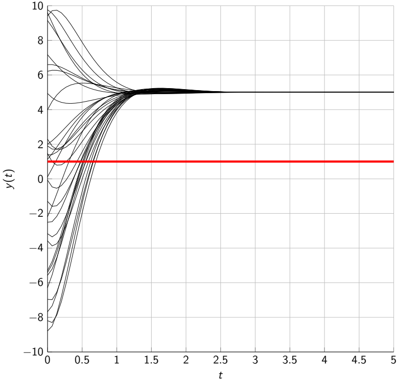
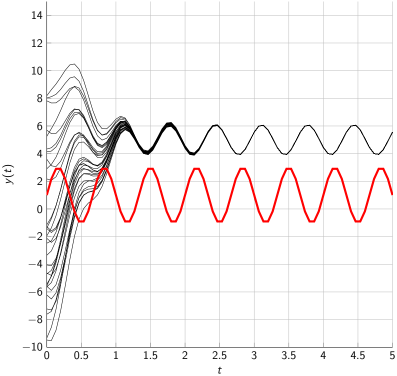
} linear systems $\downarrow$ H_∞ control

LTI: stability implies qualitative properties





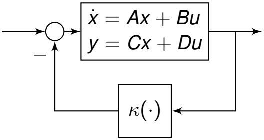
Constant response to
constant input

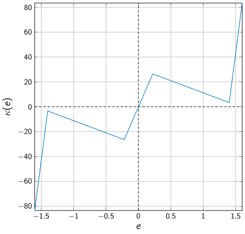


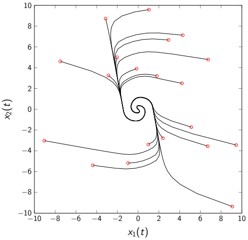
T–periodic response to
T–periodic input

Independence of initial conditions

NL: Does stability imply qualitative properties?







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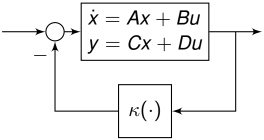
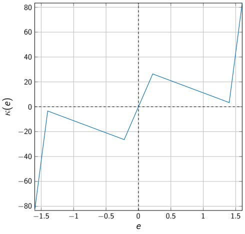
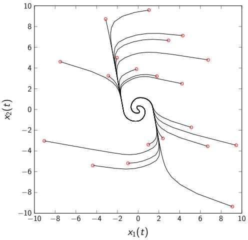
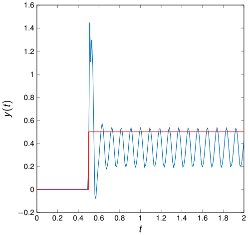






Oscillating response to constant input



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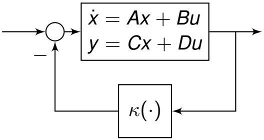
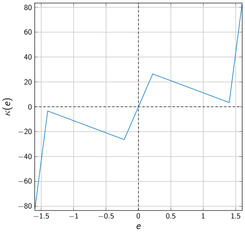
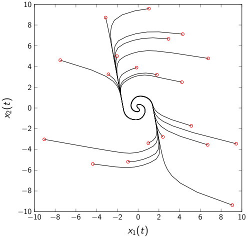
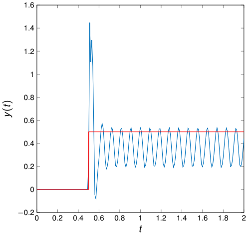






Oscillating response to constant input

Need of a stronger notion of stability



Towards nonlinear H_∞ control

LTI systems		NL systems
H_∞	$\longrightarrow$	?

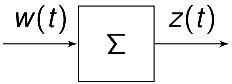
Towards nonlinear H_∞ control

LTI systems	$\longrightarrow$	NL systems
H_∞		?

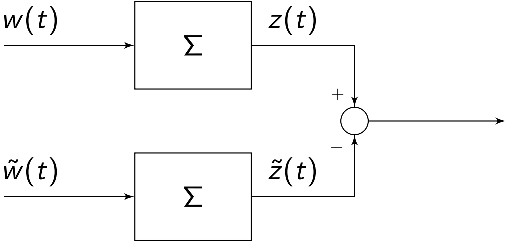
$\mathcal{L}_2$–gain proposed as a natural candidate $\rightarrow$ energetic ratio between input and output

$\mathcal{L}_2$–gain

$$\exists \gamma \; / \; \forall w : \qquad \int_0^\infty \|z(t)\|^2 \, dt \leq \gamma^2 \int_0^\infty \|w(t)\|^2 \, dt$$



	LTI	NL
$\downarrow$ Specs $\setminus$ Norm $\rightarrow$	H_∞	$\mathcal{L}_2$–gain
Constant input $\longrightarrow$ constant output	YES	NO
T periodic input $\longrightarrow$ T periodic output	YES	NO
Unique steady state	YES	NO
Convergence of the unperturbed motions	YES	NO



$\mathcal{L}_2$–gain stability is not enough $\longrightarrow$ new proposal: Incremental $\mathcal{L}_2$–gain

Incremental $\mathcal{L}_2$–gain

$$\exists \eta \;/\; \forall w, \tilde{w} : \qquad \int_0^\infty \|z(t) - \tilde{z}(t)\|^2 \, dt \leq \eta^2 \int_0^\infty \|w(t) - \tilde{w}(t)\|^2 \, dt$$

	LTI	NL	NL
↓ Specs \ Norm →	H_∞	$\mathcal{L}_2$–gain	Incremental $\mathcal{L}_2$–gain
Constant input $\longrightarrow$ constant output	YES	NO	YES
T periodic input $\longrightarrow$ T periodic output	YES	NO	YES
Unique steady state	YES	NO	YES
Convergence of the unperturbed motions	YES	NO	YES

$\mathcal{L}_2$–gain stability is not enough $\longrightarrow$ new proposal: Incremental $\mathcal{L}_2$–gain

Incremental $\mathcal{L}_2$–gain

$$\exists \eta \;/\; \forall w, \tilde{w} : \qquad \int_0^\infty \|z(t) - \tilde{z}(t)\|^2 \, dt \leq \eta^2 \int_0^\infty \|w(t) - \tilde{w}(t)\|^2 \, dt$$

Definitions

Incremental finite gain stability of a nonlinear operator $\Sigma : \mathcal{U} \longrightarrow \mathcal{Y}$

Definition

$\exists \eta \geq 0 \mid \forall w, \tilde{w} \in \mathcal{U}$:

$$\|\Sigma(w)(t) - \Sigma(\tilde{w})(t)\|_{\mathcal{Y}} \leq \eta \, \|w(t) - \tilde{w}(t)\|_{\mathcal{U}}$$

Incremental input-to-state stability of a dynamical system $\dot{x} = f(x, w)$

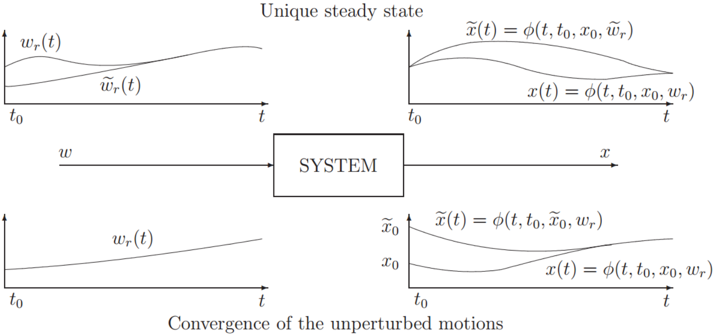
Definition

$\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty \mid \forall x_0, \tilde{x}_0, \; \forall w, \tilde{w}$:

$$\|x(t, x_0, u(t)) - \tilde{x}(t, \tilde{x}_0, \tilde{u}(t))\| \leq \beta\big(\|x_0 - \tilde{x}_0\| \, , t\big) + \gamma\big(\|w - \tilde{w}\|_\infty \big)$$

Properties

- Constant input $\rightarrow$ constant output
- T-periodic input $\rightarrow$ T-periodic output
- Unique steady state / Convergence of the unperturbed motions



Dissipativity framework

Dissipative systems

A system Σ is said to be dissipative with respect to the supply rate $s(w, z)$ if there exists a nonnegative storage function S such that

$$S(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t))\, dt \geq S(x(t_1)), \quad \forall t_1 \geq t_0 \geq 0$$

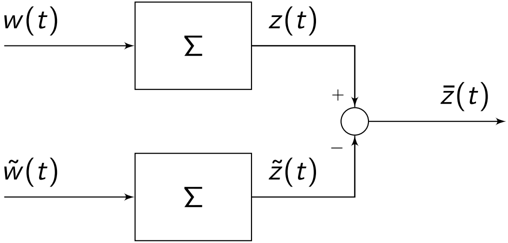
For $\mathcal{L}_2$–gain stability:

$$s(w, z) = \gamma^2 \left\| w(t) \right\|^2 - \left\| z(t) \right\|^2$$

Incremental dissipativity

Nonlinear system $\Rightarrow$ Fictitious augmented system

$$\begin{cases} \dot{x} = f(x, w) \\ z = h(x, w) \\ x(0) = x_0 \end{cases} \Rightarrow \begin{cases} \dot{x} = f(x, w) & x(0) = x_0 \\ \dot{\tilde{x}} = f(\tilde{x}, \tilde{w}) & \tilde{x}(0) = \tilde{x}_0 \\ \bar{z} = h(x, w) - h(\tilde{x}, \tilde{w}) & \end{cases}$$



$$S(x(t_0), \tilde{x}(t_0)) + \int_{t_0}^{t_1} \eta^2 \left\| w(t) - \tilde{w}(t) \right\|^2 - \left\| z(t) - \tilde{z}(t) \right\|^2 \, dt \geq S(x(t_1), \tilde{x}(t_1))$$

$$\forall t_1 \geq t_0 \geq 0$$

Sufficient conditions

Differential version

A system Σ is said to be dissipative with respect to the supply rate $s(w,z)$ if there exists a differentiable storage function S such that

$$\dot{S}(x(t),w(t)) - s(w(t),z(t)) \leq 0$$

Quadratic functions (with $P = P^T \succ 0$):

- For $\mathcal{L}_2$–gain stability:

$$S(x) = x^T P x$$

- For incremental $\mathcal{L}_2$–gain stability

$$S(x,\tilde{x}) = (x-\tilde{x})^T P (x-\tilde{x})$$

Relaxation → Sufficient conditions → Upper bound → Conservatism

↪ Piecewise Affine (PWA) representation

PWA representation

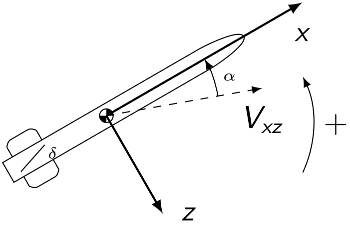
PWA regional representation

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i w(t) \\ z(t) = C_i x(t) + c_i + D_i w(t) \\ x(0) = x_0 \end{cases} \quad \text{for } x(t) \in X_i$$

Allows to:

- describe systems with saturations, relays, dead zones, etc.
- embed more generic nonlinear systems $\rightarrow$ differential inclusions
- assess performance with less conservatism

Example: Computing the $\mathcal{L}_2$–gain of NL missile



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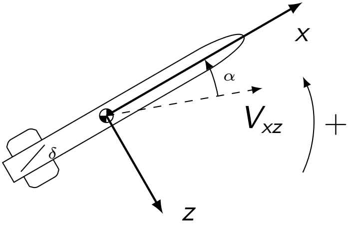

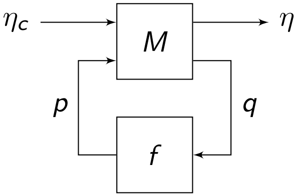

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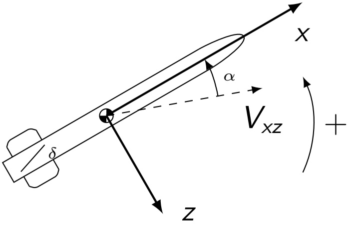

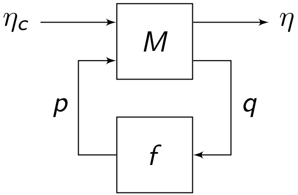

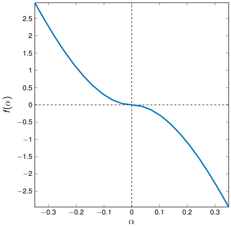

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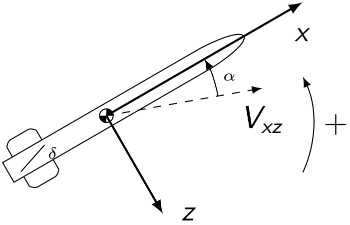

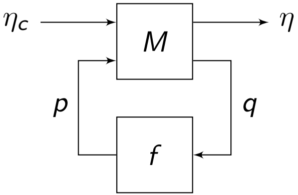

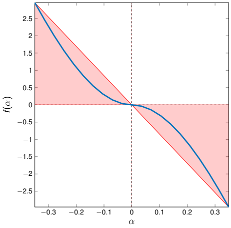

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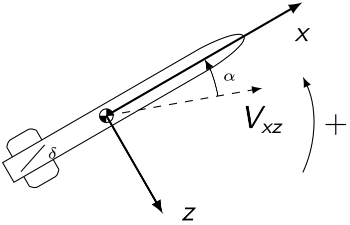

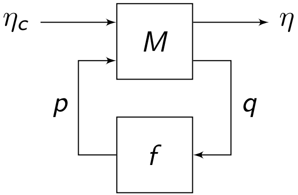

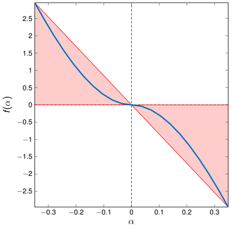

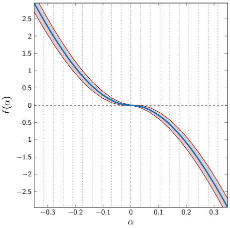

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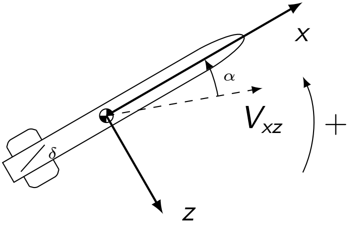

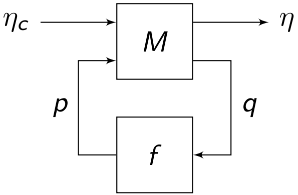

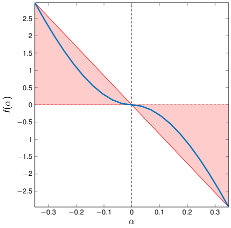

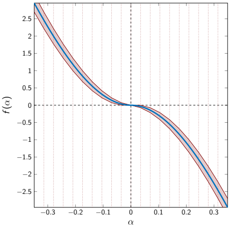

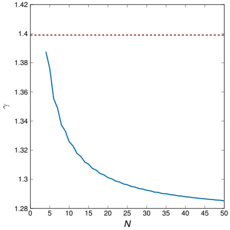

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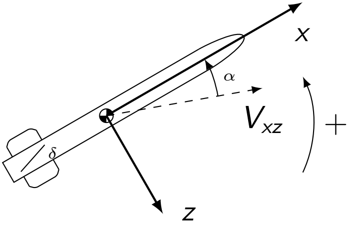
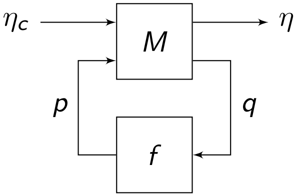
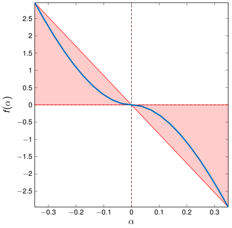
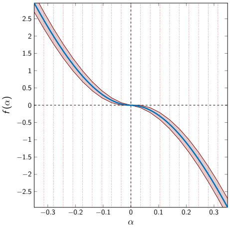
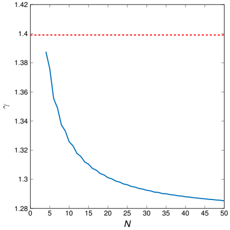








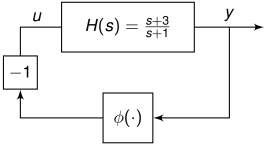


Finer computation of the upper bound to the $\mathcal{L}_2$–gain

Incremental $\mathcal{L}_2$–gain of PWA systems

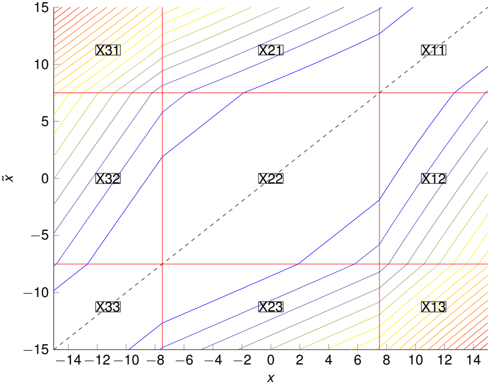
- Works of Romanchuk $\rightarrow$ Upper bound to the incremental $\mathcal{L}_2$–gain of PWA systems by means of a global quadratic function
$$S(x,\tilde{x}) = (x-\tilde{x})^T P(x-\tilde{x})$$
- Our proposal $\rightarrow$ Continuous piecewise quadratic storage functions
$$S(x,\tilde{x}) = \bar{x}^T P_{ij}\bar{x}, \text{for } \bar{x} \in X_{ij}$$
with
$$\bar{x} = \begin{bmatrix} x^T & \tilde{x}^T & 1 \end{bmatrix}^T$$
and
$$X_{ij} = \{(x,\tilde{x}) \mid x \in X_i, \tilde{x} \in X_j\}$$

One dimensional example



with

$$\phi(e) = \begin{cases} h & e > \frac{h}{\kappa} \\ \kappa e & |e| \leq \frac{h}{\kappa} \\ -h & e < -\frac{h}{\kappa} \end{cases}$$



Two dimensional example

A unique quadratic storage function assuring incremental stability does not exist

Incremental stability can be proven with piecewise quadratic storage functions