A domain decomposition strategy for solving time-harmonic Maxwell's equations discretized by a discontinuous Galerkin method

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***Abstract*— A domain decomposition strategy is introduced in order to solve time-harmonic Maxwell's equations discretized by a discontinuous Galerkin method. Its principles are explained for a 2D model problem and its efficacy is demonstrated on 2D and 3D examples.**

I. Introduction

Discontinuous Galerkin (DG) methods are emerging for the solution of time-harmonic Maxwell's equations [1] because of the enhanced flexibility compared to the conforming edge element method [2]. For instance, by using a DG method, dealing with non-conforming meshes is straightforward. Nonetheless, before taking advantage of this flexibility, the design of efficient solution algorithms has to be addressed. Here we propose a domain decomposition (DD) strategy based on optimized Schwarz methods [3].

First, the DD strategy is introduced in the two-domain case for a 2D transverse electric model problem. Then the discretization of the problem by a DG method is briefly commented. Finally, numerical results for a 2D problem confirm the expected theoretical behavior of the DD method and 3D numerical experiments on a simple geometry pave the way for more realistic applications.

II. The domain decomposition strategy

For the sake of simplicity we consider the following transverse electric model problem in a domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \text{Find the electromagnetic field } (\mathbf{E}, \mathrm{H}) \text{ satisfying:} \\ \mathrm{i}\omega\varepsilon\mathbf{E} - \mathrm{curl}\,\mathrm{H} = 0, \text{ in } \Omega, \\ \mathrm{i}\omega\mu\mathrm{H} + \mathrm{curl}\,\mathbf{E} = 0, \text{ in } \Omega, \\ \mathbf{n} \times (\mathbf{E} - \mathbf{E}^{\mathrm{inc}}) + (\mathrm{H} - \mathrm{H}^{\mathrm{inc}}) = 0, \text{ on } \partial\Omega. \end{cases} \tag{1}$$

The parameters ε and μ denote respectively the dielectric permittivity and the magnetic permeability, ω the angular frequency, $\mathbf{n}$ the unitary outgoing normal and $(\mathbf{E}^{\mathrm{inc}}, \mathrm{H}^{\mathrm{inc}})$ the components of an incident electromagnetic wave.

For solving (1), the domain Ω is decomposed in two non-overlapping subdomains Ω_1 and Ω_2. The common interface to Ω_1 and Ω_2 is denoted by Γ. The DD strategy is then a variant of the classical Schwarz method:

- We start with an initial electromagnetic field $(\mathbf{E}_l^0, \mathrm{H}_l^0)$ on each subdomain Ω_l, $l = 1, 2$.
- The $(p+1)$-th iterate $(\mathbf{E}_l^{p+1}, \mathrm{H}_l^{p+1})$ is the solution of (1) restricted to the subdomain Ω_l augmented by an interface transmission condition on Γ of the form:
$$\begin{cases} \mathbf{n} \times (\mathbf{E}_l^{p+1} - \mathbf{E}_m^p) + S_i(\mathrm{H}_l^{p+1} - \mathrm{H}_m^p) = 0, \\ \text{with } S_l = \alpha_l + \beta_l \partial_\tau^2, \end{cases} \tag{2}$$
where ∂_τ^2 denotes the second order derivative along the interface. The operator S_l ensures the transmission of the field $(\mathbf{E}_m^p, \mathrm{H}_m^p)$ computed at the previous iteration in the neighbor subdomain m with the parameters α_l and β_l properly chosen.
- The limit of the sequence $(\mathbf{E}_l^p, \mathrm{H}_l^p)_{p\in\mathbb{N}}$ is the restriction to Ω_l of $(\mathbf{E}, \mathrm{H})$ the solution of (1). Thus, we can use a stopping criterion:
$$\sum_{l=1}^{2} \frac{\|(\mathbf{E}_l^{p+1}, \mathrm{H}_l^{p+1}) - (\mathbf{E}_l^p, \mathrm{H}_l^p)\|}{\|(\mathbf{E}_l^1, \mathrm{H}_l^1) - (\mathbf{E}_l^0, \mathrm{H}_l^0)\|} < \mathrm{tol}, \tag{3}$$
where tol is the prescribed accuracy and $\|\cdot\|$ a norm.

Després in [4] was the first to propose this strategy for time-harmonic equations with the choice $S_l = 1$, for $l = 1, 2$; it coincides with a first order absorbing boundary condition. However, the convergence rate of the iterative process with this boundary condition is strongly dependent on the mesh size used for the discretization and the convergence to the solution can be slow.

Nonetheless, it is possible to greatly improve the convergence rate by optimizing it with respect to the coefficients α_l and β_l of (2). This theoretical study is done in [5] directly on (1) and in [6] for the second order $\mathrm{curl}\,\mathrm{curl}$ formulation.

The closed-form expressions obtained for the coefficients α_l and β_l are in particular dependent of the mesh size and of the size of the subdomain. These expressions are then used in a DD strategy generalized to more than two subdomains.

III. Discretization of the problem

For the discretization of the problem on Ω or on each subdomain, a DG method is used. Let us suppose that the domain is decomposed into a set of simplices T_h such that $\cup_{K\in T_h} \overline{K} = \overline{\Omega}$. The approximate solution $(\mathbf{E}_h, \mathrm{H}_h)$ of (1) is an element of V_h^3 where V_h is the finite element space of square-integrable discontinuous scalar field whose restriction to an element K is polynomial of degree k:

$$V_h = \left\{ V \in L^2(\Omega) \mid \forall K \in T_h, \quad V_{|K} \in P_k(K) \right\}. \tag{4}$$

Thus no particular continuity constraint is enforced at the interfaces between elements. In order to keep the consistency of the discretization, some numerical flux has to be defined on these interfaces weakly enforcing the continuity of the components of the electric and magnetic fields tangential to the interface [7].

IV. Numerical results

A. Two-dimensional problem

A theoretical convergence rate ρ depending on the mesh size h is deduced from the analysis in [5]. This analysis is done on the continuous, *i.e.* without discretization, DD method but we can check that the results remain valid with a discretization. In the 2D case, two particular choices of some parameters p_1 and p_2, leading to a concise expression of ρ, are reported in Tab. I and then α_l is equal to $(\mathrm{i}\omega)^{-1}(p_l + \mathrm{i}p_l)$ and β_l is zero. The coefficient C_ω in Tab. I is set for a given ω; a precise definition of its role can be found in [5].

TABLE I

CONVERGENCE RATE AND TRANSMISSION CONDITION PARAMETERS.

Case	ρ	p_1	p_2
A	$1-\frac{\sqrt{2}C_\omega^{\frac{1}{4}}}{\sqrt{\pi}}\sqrt{h}$	$\frac{\sqrt{\pi}C_\omega^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$	$\frac{\sqrt{\pi}C_\omega^{\frac{1}{4}}}{\sqrt{2}\sqrt{h}}$
B	$1-\frac{C_\omega^{\frac{1}{8}}}{\pi^{\frac{1}{4}}}h^{\frac{1}{4}}$	$\frac{\pi^{\frac{1}{4}}C_\omega^{\frac{1}{8}}}{2h^{\frac{1}{4}}}$	$\frac{\pi^{\frac{3}{4}}C_\omega^{\frac{1}{8}}}{h^{\frac{3}{4}}}$

The agreement between the theoretical and numerical convergence rates is demonstrated on a problem with $\Omega =]0;1[^2$, $(\mathbf{E}_x^{\mathrm{inc}}, \mathbf{E}_y^{\mathrm{inc}}, \mathrm{H}^{\mathrm{inc}}) = \exp(-\mathrm{i}\omega x)(0,1,1)$ and $\omega = 2\pi$. The DG discretization is based on a triangular uniform mesh with $P_1(K)$ as the local space in (4). On Fig. 1, the number of iterations for achieving a prescribed accuracy against the mesh size is shown for both boundary conditions (Case A and Case B). The curves fit nicely the dependence in h predicted by the theory *i.e.* it behaves like $h^{-0.5}$ for Case A and like $h^{-0.25}$ for Case B.

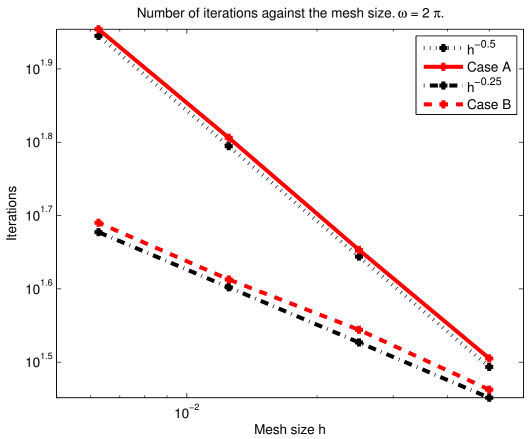

Fig. 1. Number of iterations against the mesh size h. Logarithmic scale.

B. Three-dimensional problem

The implementation of optimized interface conditions for 3D time-harmonic Maxwell's equations is a work in progress. Here, we give preliminary results for the DD strategy based on first order absorbing boundary conditions as transmission conditions.

The problem under consideration is the scattering of a plane wave by a perfectly conducting unit sphere. The incident wave is given by $\mathbf{E}^{\mathrm{inc}} = (\exp(-i\omega x), 0, 0)$ and $\mathbf{H}^{\mathrm{inc}} = (0, \exp(-i\omega x), 0)$, with $\omega = 4\pi$. The absorbing boundary is set to one wavelength from the surface of the perfectly conducting sphere. The mesh is composed of 1,382,400 tetrahedra and a $P_0(K)$ local space is used for the DG method. The total number of unknowns is 8,294,400.

Numerical experiments are conducted on a cluster of 64 AMD Opteron/2 GHz processors with a Gigabit Ethernet interconnection. One subdomain is associated to each processor and a sparse matrix direct method is used to solve the subdomain problem. As it is explained in [3], the DD method can be formulated as a linear system whose unknowns are auxiliary interface variables. This interface system is usually solved by a Krylov method which gives more robustness to the DD strategy. Here we make use of a BiCGstab(l) Krylov method [8] either for solving the interface system or as a global solver without preconditioner.

Performance results are given in Tab. II where 'DDM' refers to the DD solution strategy. The time per processor for performing the factorization is 18.0 sec (min)/102.0 sec (max) while the associated memory usage is 405 MB (min)/1001 MB (max). In addition to the gain in computing time, a clear advantage of the DD strategy is its parallel efficacy that can be evaluated here as the ratio of 'CPU (max)' over 'Elapsed' which is equal to 92% while the corresponding feature for the global solver is 74%.

TABLE II

PERFORMANCE RESULTS. 'CPU (MIN/MAX)' ARE MEASURES PER PROCESSOR OF THE CPU TIME. 'ELAPSED' IS THE ELAPSED TIME.

Solver	CPU (min)	CPU (max)	Elapsed
Global	1940.0 sec	2142.0 sec	2919.0 sec
DDM	259.0 sec	413.0 sec	449.0 sec

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